4.4 The MVT for integrals (average value over an interval)

and the

1st & 2nd FTC

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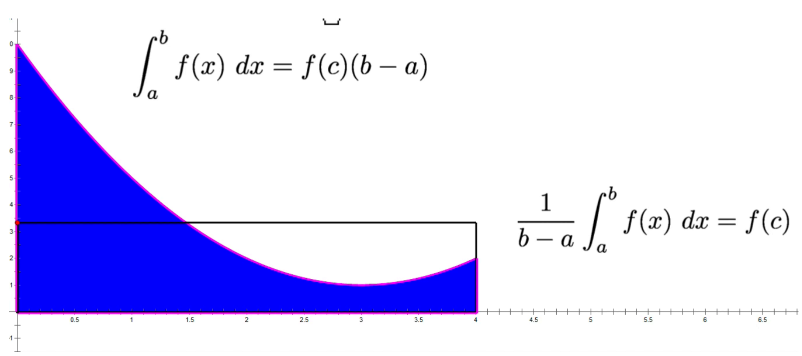
Average Value on a Interval

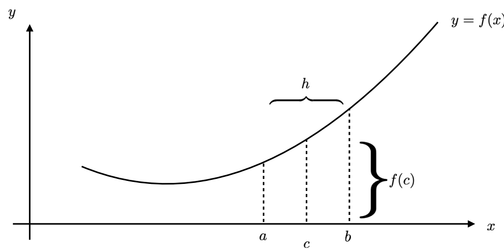
Mean Value Theorem for Integrals



(Thanks Jim O'Connor for this Animation!)

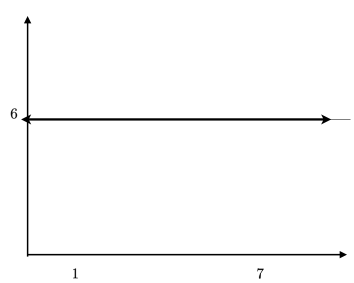
There must be a c in $[a, b]$ where

$$f(c) \cdot (b-a) = \int_a^b f(x)\ dx$$



$f(c)$ is the average value of f on the interval $[a, b]$

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\ dx$$



$$f(x) = 6$$

Using Geometry..

$$\int_1^7 f(x)\ dx \quad = (7-1)(6) = 36$$

$$\frac{1}{7-1}\int_1^7 f(x)\ dx = \frac{36}{6} = 6$$

Using Calculus..

$$\int f(x)\ dx = F(x) = 6x + C$$

$$\text{If } F(0) = 0 \text{ then } C = 0$$

$$F(x) = 6x \text{ and } F'(x) = f(x) = 6$$

Notice this..

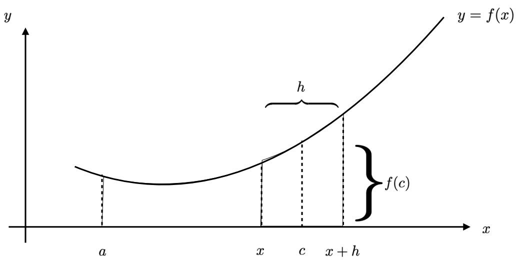
$$F(7) = 6(7) = 42$$

$$F(1) = 6(1) = 6$$

$$42 - 6 = 36 = \int_1^7 f(x)\ dx \quad = F(7) - F(1)$$ **Will this ALWAYS work?**

Claim

$$\int_a^b f(x)\ dx = F(b) - F(a)$$



$A(x) = \int_a^x f(t)\ dt$ **... and ...** $\int f(x)\ dx = F(x) + C$, if $F'(x) = f(x)$

$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h\to 0} \frac{f(c)\cdot \cancel{h}}{\cancel{h}} = \lim_{h\to 0} f(c) = f(x)$ so $A(x)$ is an antiderivative of $f(x)$

$A(x) = F(x) + C = \int_a^x f(t)\ dt$ consider $A(a) = \int_a^a f(t)\ dt = 0 = F(x) + C$

Here $x = a$ so $F(a) + C = 0$

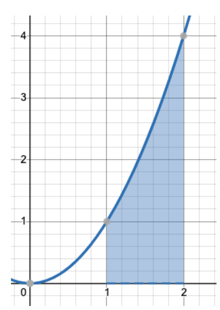
Then $\int_a^x f(t)\ dt = F(x) - F(a)$ ... so $C = -F(a)$

If we let $x = b$ we have the FTC: $\int_a^b f(x)\ dx = F(b) - F(a)$

4.4 The First Fundamental Theorem of Calculus

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

$$F(a) + \int_a^b f(x)\ dx = F(b)$$



$\int_1^2 x^2\ dx$ So $F(x) = \frac{x^3}{3}$

$$= \left.\frac{x^3}{3}\right|_1^2 = F(2) - F(1)$$

$$= \frac{2^3}{3} - \frac{1^3}{3}$$

$$= \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

The Net Change "Theorem"

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

$$F(a) + \int_a^b f(x)\ dx = F(b)$$

A tank contains 10 gallons. Water is added at a rate of 4 gallons per minute, but leaks at $\sqrt{t}$ gallons per minute for time $t \geq 0$. How much is in the tank after 30 minutes?

$$10 + \int_0^{30} 4 - t^{1/2}\ dt$$

$$10 + 4t - \frac{2}{3}t^{3/2}\Big|_0^{30}$$

$$10 + 4(30) - \frac{2}{3}30^{3/2}\Big|_0^{30}$$

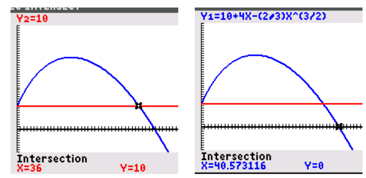
$$\approx 20.455 \text{ gallons}$$

How long will it take to have 10 gallons again?
How long will it take to be empty?

$$Y_1 = 10 + \left(4(t) - \frac{2}{3}t^{3/2}\right)$$

$$Y_2 = 10$$

$$Y_3 = 0$$



The tank will have 10 gallons again after 36 minutes and be empty in 45.573 minutes.

The Second Fundamental Theorem of Calculus

$$\frac{d}{dx}\left[\int f(x)\ dx\right] = f(x)$$

Or the "Chain Rule" Version:

$$\frac{d}{dx}\left[\int_a^{g(x)} f(t)\ dt\right] = f(g(x)) \cdot g'(x)$$

$$\frac{d}{dx}\int_2^x \tan(t^3)\ dt = \ \tan x^3$$

$$\frac{d}{dx}\int_{-1}^{x^3} \frac{1}{1+t}\ dt = \frac{1}{1+x^3} \cdot 3x^2 = \frac{3x^2}{1+x^3}$$

$$\frac{d}{dx}\int_2^{\sin x} \sqrt[3]{1+t^2}\ dt = \sqrt[3]{1+\sin^2 x} \cdot \cos x$$

$$\frac{d}{dy}\int_\pi^{3y} 14x^2\ dx = 14(3y)^2 \cdot 3 = 378y^2$$